Contents 18

Functions of Several Variables

Learning outcomes

In this Workbook you will learn about functions of two or more variables. You will learn that a function of two variables can be interpreted as a surface. You will learn how to sketch simple surfaces. You will learn what a partial derivative is and how the partial derivative of any order may be found. As an important application of partial differentiation you will learn how to locate the turning points of functions of several variables. In particular, for functions of two variables, you will learn how to distinguish between maxima and minima points. Finally you will apply your knowledge to the topic of error analysis.

Functions of Several Variables

18.1

Introduction

A function of a single variable $y = f(x)$ is interpreted graphically as a planar curve. In this Section we generalise the concept to functions of more than one variable. We shall see that a function of two variables $z = f(x, y)$ can be interpreted as a surface. Functions of two or more variables often arise in engineering and in science and it is important to be able to deal with such functions with confidence and skill. We see in this Section how to sketch simple surfaces. In later Sections we shall examine how to determine the rate of change of $f(x, y)$ with respect to x and y and also how to obtain the optimum values of functions of several variables.

Prerequisites

Before starting this Section you should ...

- understand the Cartesian coordinates (x, y, z) of three-dimensional space.
- be able to sketch simple 2D curves

Learning Outcomes

On completion you should be able to ...

- understand the mathematical description of a surface
- sketch simple surfaces
- use the notation for a function of several variables

1. Functions of several variables

We know that $f(x)$ is used to represent a function of one variable: the input variable is x and the output is the value $f(x)$. Here x is the **independent** variable and $y = f(x)$ is the **dependent variable**.

Suppose we consider a function with **two independent** input variables x and y, for example

$$f(x, y) = x + 2y + 3.$$

If we specify values for x and y then we have a **single** value $f(x, y)$. For example, if $x = 3$ and $y = 1$ then $f(x, y) = 3 + 2 + 3 = 8$. We write $f(3, 1) = 8$.

Task

Find the values of $f(2, 1)$, $f(-1, -3)$ and $f(0, 0)$ for the following functions.

(a) $f(x, y) = x^2 + y^2 + 1$ (b) $f(x, y) = 2x + xy + y^3$

Your solution

Answer

(a) $f(2, 1) = 2^2 + 1^2 + 1 = 6$; $f(-1, -3) = (-1)^2 + (-3)^2 + 1 = 11$; $f(0, 0) = 1$

(b) $f(2, 1) = 4 + 2 + 1 = 7$; $f(-1, -3) = -2 + 3 - 27 = -26$; $f(0, 0) = 0$

In a similar way we can define a function of three independent variables. Let these variables be x, y and u and the function $f(x, y, u)$.

Example 1

Given $f(x, y, u) = x^2 + yu + 2$, find $f(0, 1, 0)$, $f(-1, -1, 2)$.

Solution

$$f(0, 1, 0) = 0^2 + 1 \times 0 + 2 = 2; \quad f(-1, -1, 2) = 1 - 2 + 2 = 1$$

Task

(a) Find $f(2, -1, 1)$ for $f(x, y, u) = xy + yu + ux$.

(b) Evaluate $f(x, y, u, t) = x^2 - y^2 - u^2 - 2t$ when $x = 1$, $y = -2$, $u = 3$, $t = 1$.

Your solution

Answer
(a) $f(2,-1,1) = 2 \times (-1) + (-1) \times 1 + 1 \times 2 = -1$
(b) $f(1,-2,3,1) = 1^2 - (-2)^2 - 3^2 - 2 \times 1 = -14$ (this is a function of 4 independent variables).

2. Functions of two variables

The aim of this Section is to enable the reader to gain confidence in dealing with functions of several variables. In order to do this we often concentrate on functions of just two variables. The latter have an easy geometrical interpretation and we can therefore use our geometrical intuition to help understand the meaning of much of the mathematics associated with such functions. We begin by reminding the reader of the Cartesian coordinate system used to locate points in three dimensions. A point P is located by specifying its Cartesian coordinates (a, b, c) defined in Figure 1.

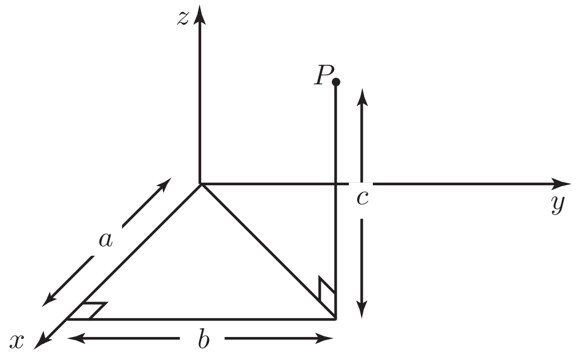


Figure 1

Within this 3-dimensional space we can consider simple surfaces. Perhaps the simplest is the **plane**. From HELM 9.6 on vectors we recall the general equation of a plane:

$$Ax + By + Cz = D$$

where $A,\ B,\ C,\ D$ are constants. This plane intersects the $x-$axis (where $y = z = 0$) at the point $\left(\dfrac{D}{A},\ 0,\ 0\right)$, intersects the $y-$axis (where $x = z = 0$) at the point $\left(0,\ \dfrac{D}{B}, 0\right)$ and the $z-$axis (where $x = y = 0$) at the point $\left(0,\ 0,\ \dfrac{D}{C}\right)$. See Figure 2 where the dotted lines are hidden from view behind the plane which passes through three points marked on the axes.

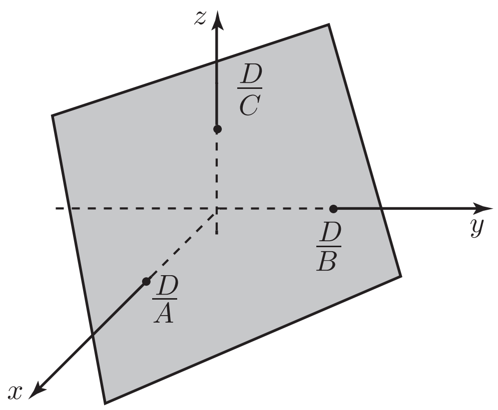


Figure 2

There are some special cases of note.

- $B = C = 0 \quad A \neq 0.$

  Here the plane is $x = D/A$. This plane (for any given values of D and A) is parallel to the zy plane a distance D/A units from it. See Figure 3a.

- $A = 0,\ C = 0 \quad B \neq 0$

  Here the plane is $y = D/B$ and is parallel to the zx plane at a distance D/B units from it. See Figure 3b.

- $A = 0,\ B = 0 \quad C \neq 0$

  Here the plane is $z = D/C$ which is parallel to the xy plane a distance D/C units from it. See Figure 3c.

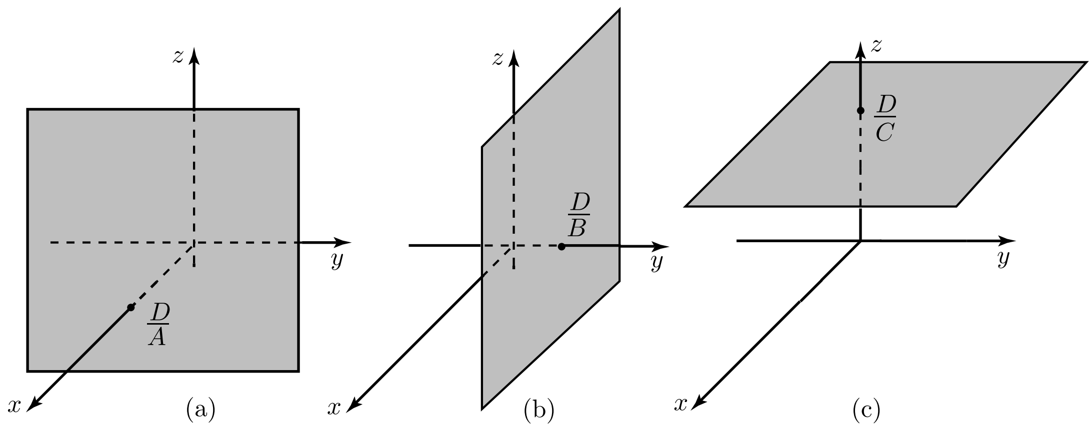


Figure 3

Planes are particularly simple examples of **surfaces**. Generally, a surface is described by a relation connecting the three variables x, y, z. In the case of the plane this relation is linear $Ax+By+Cz = D$. In some cases, as we have seen, one or two variables may be absent from the relation. In three dimensions such a relation **still** defines a surface, for example $z = 0$ defines the plane of the x- and y-axes.

Although **any** relation connecting x, y, z defines a surface, by convention, one of the variables (usually z) is chosen as the dependent variable and the other two therefore are independent variables. For the case of a plane $Ax + By + Cz = D$ (and $C \neq 0$) we would write, for example,

$$z = \frac{1}{C}(D - Ax - By)$$

Generally a surface is defined by a relation of the form

$$z = f(x, y)$$

where the expression on the right is any relation involving two variables x, y.

Sketching surfaces

A plane is relatively easy to sketch since it is flat all we need to know about it is where it intersects the three coordinate axes. For more general surfaces what we do is to sketch curves (like contours) which lie on the surface. If we draw enough of these curves our 'eye' will naturally interpret the shape of the surface.

Let us see, for example, how we sketch $z = x^2 + y^2$.
Firstly we confirm that $z = x^2+y^2$ **is a surface** since this is a relation connecting the three coordinate variables x, y, z. In the standard notation our function of two variables is $f(x, y) = x^2+y^2$. To sketch the surface we fix one of the variables at a constant value.

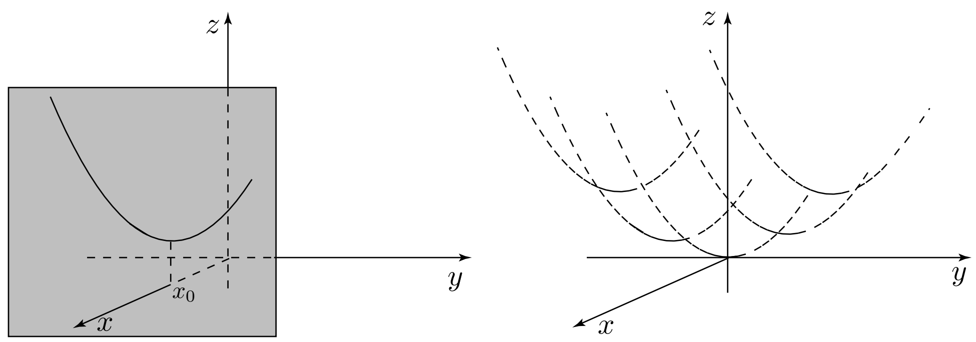
- Fix x at value x_0.

From our discussion above we remember that $x = x_0$ is the equation of a plane parallel to the zy plane. In this case our relation becomes:

$$z = x_0^2 + y^2$$

Since z is now a function of a **single** variable y, with x_0^2 held constant, this relation: $z = x_0^2 + y^2$ defines a **curve** which **lies in the plane** $x = x_0$.

In Figure 4(a) we have drawn this curve (a **parabola**). Now by changing the value chosen for x_0 we will obtain a sequence of curves, each a parabola, lying in a different plane, and each being a part of the surface we are trying to sketch. In Figure 4(b) we have drawn some of the curves of this sequence.



(a) (b)

Figure 4

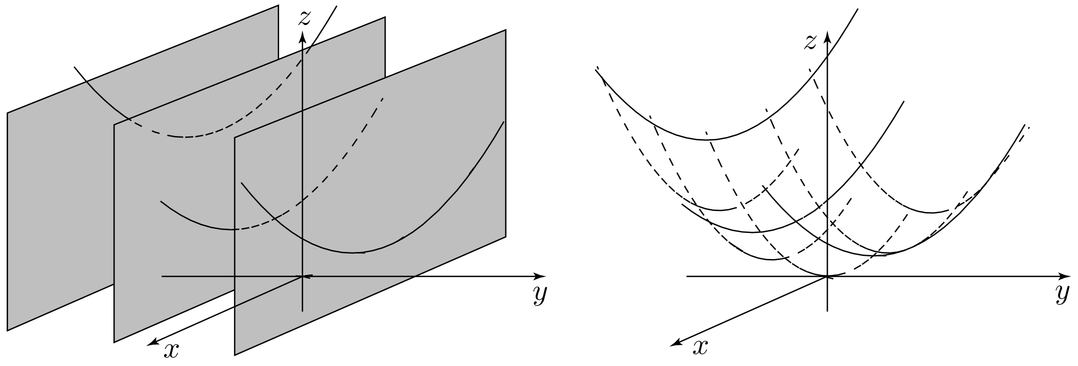
What we have done is to **slice** the surface by planes parallel to the zy plane. Each slice intersects the surface in a curve. In this case we have not yet plotted enough curves to accurately visualise the surface so we need to draw other surface curves.

- Fix y at value y_0

Here $y = y_0$ (the equation of a plane parallel to the zx plane.) In this case the surface becomes

$$z = x^2 + y_0^2$$

Again z is a function of single variable (since y_0 is fixed) and describes a curve: again the curve is a **parabola**, but this time residing on the plane $y = y_0$. For each different y_0 we choose a different parabola is obtained: each lying on the surface $z = x^2+y^2$. Some of these curves have been sketched in Figure 5(a). These have then combined with the curves of Figure 4(b) to produce Figure 5(b).



(a) (b)

Figure 5

We now have an idea of what the surface defined by $z = x^2 + y^2$ looks like but to complete the picture we draw a final sequence of curves.

- Fix z at value z_0.

We have $z = z_0$ (the equation of a plane parallel to the xy plane.) In this case the surface becomes

$$z_0 = x^2 + y^2$$

But this is the equation of a **circle** centred on $x = 0$, $y = 0$ of radius $\sqrt{z_0}$. (Clearly we **must** choose $z_0 \geq 0$ because $x^2 + y^2$ cannot be negative.) As we vary z_0 we obtain different circles, each lying on a different plane $z = z_0$. In Figure 6 we have combined the circles with the curves of Figure 5(b) to obtain a good visualisation of the surface $z = x^2 + y^2$.

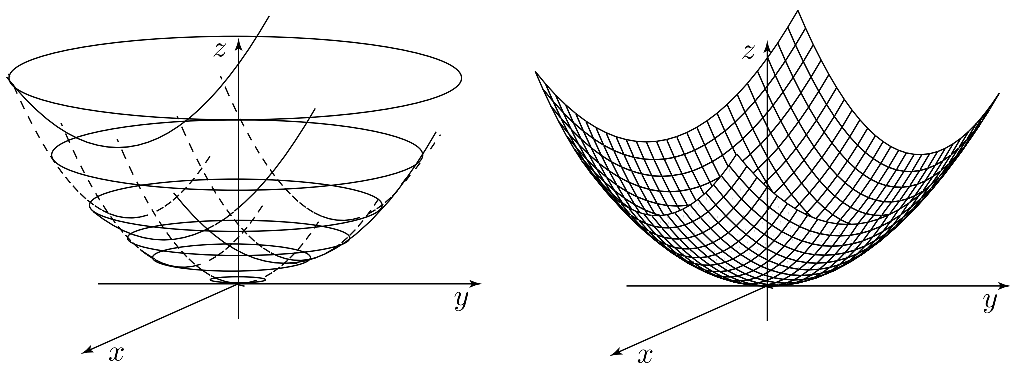


Figure 6

(Technically the surface is called a **paraboloid**, obtained by **rotating** a parabola about the $z-$axis.)

With the wide availability of sophisticated graphics packages the need to be able to sketch a surface is not as important as once it was. However, we urge the reader to attempt simple surface sketching in the initial stages of this study as it will enhance understanding of functions of two variables.

Partial Derivatives

18.2

Introduction

When a function of more than one independent input variable changes because of changes in one or more of the input variables, it is important to calculate the change in the function itself. This can be investigated by holding all but one of the variables constant and finding the rate of change of the function with respect to the one remaining variable. This process is called partial differentiation. In this Section we show how to carry out the process.

Prerequisites

Before starting this Section you should ...

- understand the principle of differentiating a function of one variable

Learning Outcomes

On completion you should be able to ...

- understand the concept of partial differentiation
- differentiate a function partially with respect to each of its variables in turn
- evaluate first partial derivatives
- carry out successive partial differentiations
- formulate second partial derivatives

1. First partial derivatives

The x partial derivative

For a function of a single variable, $y = f(x)$, changing the independent variable x leads to a corresponding change in the dependent variable y. The **rate of change** of y with respect to x is given by the **derivative**, written $\dfrac{df}{dx}$. A similar situation occurs with functions of more than one variable. For clarity we shall concentrate on functions of just two variables.

In the relation $z = f(x, y)$ the **independent** variables are x and y and the **dependent variable** z. We have seen in Section 18.1 that as x and y vary the z-value traces out a surface. Now both of the variables x and y may change *simultaneously* inducing a change in z. However, rather than consider this general situation, to begin with we shall hold one of the independent variables **fixed**. This is equivalent to moving along a curve obtained by intersecting the surface by one of the coordinate planes.

Consider $f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$.

Suppose we keep y constant and vary x; then what is the rate of change of the function f?

Suppose we hold y at the value 3 then

$$f(x, 3) = x^3 + 6x^2 + 9 + 2x + 1 = x^3 + 6x^2 + 2x + 10$$

In effect, we now have a function of x only. If we differentiate it with respect to x we obtain the expression:

$$3x^2 + 12x + 2.$$

We say that f has been **partially differentiated** with respect to x. We denote the partial derivative of f with respect to x by $\dfrac{\partial f}{\partial x}$ (to be read as 'partial dee f by dee x'). In this example, when $y = 3$:

$$\frac{\partial f}{\partial x} = 3x^2 + 12x + 2.$$

In the same way if y is held at the value 4 then $f(x, 4) = x^3 + 8x^2 + 16 + 2x + 1 = x^3 + 8x^2 + 2x + 17$ and so, for this value of y

$$\frac{\partial f}{\partial x} = 3x^2 + 16x + 2.$$

Now if we return to the original formulation

$$f(x, y) = x^3 + 2x^2y + y^2 + 2x + 1$$

and treat y as a constant then the process of partial differentiation with respect to x gives

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2 + 4xy + 0 + 2 + 0 \\ &= 3x^2 + 4xy + 2.\end{aligned}$$

Key Point 1

The Partial Derivative of f with respect to x

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to x is denoted by $\dfrac{\partial f}{\partial x}$ and is obtained by differentiating $f(x, y)$ with respect to x in the usual way but treating the y-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial x}$ are $f_x(x, y)$ or f_x or $\dfrac{\partial z}{\partial x}$.

Example 2

Find $\dfrac{\partial f}{\partial x}$ for (a) $f(x, y) = x^3 + x + y^2 + y$, (b) $f(x, y) = x^2y + xy^3$.

Solution

(a) $\dfrac{\partial f}{\partial x} = 3x^2 + 1 + 0 + 0 = 3x^2 + 1$ (b) $\dfrac{\partial f}{\partial x} = 2x \times y + 1 \times y^3 = 2xy + y^3$

The y partial derivative

For functions of two variables $f(x, y)$ the x and y variables are on the same footing, so what we have done for the x-variable we can do for the y-variable. We can thus imagine keeping the x-variable fixed and determining the rate of change of f as y changes. This rate of change is denoted by $\dfrac{\partial f}{\partial y}$.

Key Point 2

The Partial Derivative of f with respect to y

For a function of two variables $z = f(x, y)$ the partial derivative of f with respect to y is denoted by $\dfrac{\partial f}{\partial y}$ and is obtained by differentiating $f(x, y)$ with respect to y in the usual way but treating the x-variable as if it were a constant.

Alternative notations for $\dfrac{\partial f}{\partial y}$ are $f_y(x, y)$ or f_y or $\dfrac{\partial z}{\partial y}$.

Returning to $f(x,y) = x^3 + 2x^2y + y^2 + 2x + 1$ once again, we therefore obtain:

$$\frac{\partial f}{\partial y} = 0 + 2x^2 \times 1 + 2y + 0 + 0 = 2x^2 + 2y.$$

Example 3

Find $\dfrac{\partial f}{\partial y}$ for (a) $f(x,y) = x^3 + x + y^2 + y$ (b) $f(x,y) = x^2y + xy^3$

Solution

(a) $\dfrac{\partial f}{\partial y} = 0 + 0 + 2y + 1 = 2y + 1$ (b) $\dfrac{\partial f}{\partial y} = x^2 \times 1 + x \times 3y^2 = x^2 + 3xy^2$

We can calculate the partial derivative of f with respect to x and the value of $\dfrac{\partial f}{\partial x}$ at a specific point e.g. $x = 1,\ y = -2$.

Example 4

Find $f_x(1,-2)$ and $f_y(-3,2)$ for $f(x,y) = x^2 + y^3 + 2xy$.

[Remember f_x means $\dfrac{\partial f}{\partial x}$ and f_y means $\dfrac{\partial f}{\partial y}$.]

Solution

$f_x(x,y) = 2x + 2y$, so $f_x(1,-2) = 2 - 4 = -2$; $f_y(x,y) = 3y^2 + 2x$, so $f_y(-3,2) = 12 - 6 = 6$

Task

Given $f(x,y) = 3x^2 + 2y^2 + xy^3$ find $f_x(1,-2)$ and $f_y(-1,-1)$.

First find expressions for $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\dfrac{\partial f}{\partial y} =$

Answer

$\dfrac{\partial f}{\partial x} = 6x + y^3,\quad \dfrac{\partial f}{\partial y} = 4y + 3xy^2$

Now calculate $f_x(1,-2)$ and $f_y(-1,-1)$:

Your solution

$f_x(1,-2) =$ $\qquad\qquad$ $f_y(-1,-1) =$

Answer

$f_x(1,-2) = 6\times 1 + (-2)^3 = -2;\qquad f_y(-1,-1) = 4\times(-1) + 3(-1)\times 1 = -7$

Functions of several variables

As we have seen, a function of two variables $f(x,y)$ has two partial derivatives, $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$. In an exactly analogous way a function of three variables $f(x,y,u)$ has three partial derivatives $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial u}$, and so on for functions of more than three variables. Each partial derivative is obtained in the same way as stated in Key Point 3:

Key Point 3

The Partial Derivatives of $f(x,y,u,v,w,\ldots)$

For a function of several variables $z = f(x,y,u,v,w,\ldots)$ the partial derivative of f with respect to v (say) is denoted by $\dfrac{\partial f}{\partial v}$ and is obtained by differentiating $f(x,y,u,v,w,\ldots)$ with respect to v in the usual way but treating all the other variables as if they were constants.

Alternative notations for $\dfrac{\partial f}{\partial v}$ when $z = f(x,y,u,v,w,\ldots)$ are $f_v(x,y,u,v,w\ldots)$ and f_v and $\dfrac{\partial z}{\partial v}$.

Task

Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial u}$ for $f(x,y,u,v) = x^2 + xy^2 + y^2u^3 - 7uv^4$

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial u} =$

Answer

$\dfrac{\partial f}{\partial x} = 2x + y^2 + 0 + 0 = 2x + y^2;\qquad \dfrac{\partial f}{\partial u} = 0 + 0 + y^2\times 3u^2 - 7v^4 = 3y^2u^2 - 7v^4.$

Task

The pressure, P, for one mole of an ideal gas is related to its absolute temperature, T, and specific volume, v, by the equation

$$Pv = RT$$

where R is the gas constant.

Obtain simple expressions for

(a) the coefficient of thermal expansion, α, defined by:

$$\alpha = \frac{1}{v}\left(\frac{\partial v}{\partial T}\right)_P$$

(b) the isothermal compressibility, κ_T, defined by:

$$\kappa_T = -\frac{1}{v}\left(\frac{\partial v}{\partial P}\right)_T$$

Your solution

(a)

Answer

$$v = \frac{RT}{P} \quad \Rightarrow \quad \left(\frac{\partial v}{\partial T}\right)_P = \frac{R}{P}$$

so $\quad \alpha = \dfrac{1}{v}\left(\dfrac{\partial v}{\partial T}\right)_P = \dfrac{R}{Pv} = \dfrac{1}{T}$

Your solution

(b)

Answer

$$v = \frac{RT}{P} \quad \Rightarrow \quad \left(\frac{\partial v}{\partial P}\right)_T = -\frac{RT}{P^2}$$

so $\quad \kappa_T = -\dfrac{1}{v}\left(\dfrac{\partial v}{\partial P}\right)_T = \dfrac{RT}{vP^2} = \dfrac{1}{P}$

Exercises

1. For the following functions find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

   (a) $f(x,y) = x + 2y + 3$
   (b) $f(x,y) = x^2 + y^2$
   (c) $f(x,y) = x^3 + xy + y^3$
   (d) $f(x,y) = x^4 + xy^3 + 2x^3y^2$
   (e) $f(x,y,z) = xy + yz$

2. For the functions of Exercise 1 (a) to (d) find $f_x(1,1),\ f_x(-1,-1),\ f_y(1,2),\ f_y(2,1)$.

Answers

1. (a) $\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = 2$

   (b) $\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y$

   (c) $\frac{\partial f}{\partial x} = 3x^2 + y, \quad \frac{\partial f}{\partial y} = x + 3y^2$

   (d) $\frac{\partial f}{\partial x} = 4x^3 + y^3 + 6x^2y^2, \quad \frac{\partial f}{\partial y} = 3xy^2 + 4x^3y$

   (e) $\frac{\partial f}{\partial x} = y, \quad \frac{\partial f}{\partial y} = x + z$

2.

	$f_x(1,1)$	$f_x(-1,-1)$	$f_y(1,2)$	$f_y(2,1)$
(a)	1	1	2	2
(b)	2	-2	4	2
(c)	4	2	13	5
(d)	11	1	20	38

2. Second partial derivatives

Performing two successive partial differentiations of $f(x,y)$ with respect to x (holding y constant) is denoted by $\dfrac{\partial^2 f}{\partial x^2}$ (or $f_{xx}(x,y)$) and is defined by

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

For functions of two or more variables as well as $\dfrac{\partial^2 f}{\partial x^2}$ other second-order partial derivatives can be obtained. Most obvious is the second derivative of $f(x,y)$ with respect to y is denoted by $\dfrac{\partial^2 f}{\partial y^2}$ (or $f_{yy}(x,y)$) which is defined as:

$$\frac{\partial^2 f}{\partial y^2} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

Example 5

Find $\dfrac{\partial^2 f}{\partial x^2}$ and $\dfrac{\partial^2 f}{\partial y^2}$ for $f(x,y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 + 0 + 2 + 0 = 3x^2 + 2xy^2 + 2$$

$$\frac{\partial^2 f}{\partial x^2} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = 6x + 2y^2 + 0 = 6x + 2y^2.$$

$$\frac{\partial f}{\partial y} = 0 + x^2 \times 2y + 6y^2 + 0 + 1 = 2x^2y + 6y^2 + 1$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = 2x^2 + 12y.$$

We can use the alternative notation when evaluating derivatives.

Example 6

Find $f_{xx}(-1,1)$ and $f_{yy}(2,-2)$ for $f(x,y) = x^3 + x^2y^2 + 2y^3 + 2x + y$.

Solution

$f_{xx}(-1,1) = 6 \times (-1) + 2 \times (-1)^2 = -4.$

$f_{yy}(2,-2) = 2 \times (2)^2 + 12 \times (-2) = -16$

Mixed second derivatives

It is possible to carry out a partial differentiation of $f(x, y)$ with respect to x followed by a partial differentiation with respect to y (or vice-versa). The results are examples of **mixed derivatives**. We must be careful with the notation here.

We use $\dfrac{\partial^2 f}{\partial x \partial y}$ to mean 'differentiate first with respect to y and then with respect to x' and we use $\dfrac{\partial^2 f}{\partial y \partial x}$ to mean 'differentiate first with respect to x and then with respect to y':

$$\text{i.e.} \qquad \frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \frac{\partial^2 f}{\partial y \partial x} \equiv \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

(This explains why the order is opposite of what we expect - the derivative 'operates on the left'.)

Example 7

For $f(x, y) = x^3 + 2x^2y^2 + y^3$ find $\dfrac{\partial^2 f}{\partial x \partial y}$.

Solution

$\dfrac{\partial f}{\partial y} = 4x^2y + 3y^2$; $\quad \dfrac{\partial^2 f}{\partial x \partial y} = 8xy$

The remaining possibility is to differentiate first with respect to x and then with respect to y i.e. $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

For the function in Example 7 $\dfrac{\partial f}{\partial x} = 3x^2 + 4xy^2$ and $\dfrac{\partial^2 f}{\partial y \partial x} = 8xy$. Notice that for this function

$$\frac{\partial^2 f}{\partial x \partial y} \equiv \frac{\partial^2 f}{\partial y \partial x}.$$

This equality of mixed derivatives is true for all functions which you are likely to meet in your studies.

To evaluate a mixed derivative we can use the alternative notation. To evaluate $\dfrac{\partial^2 f}{\partial x \partial y}$ we write $f_{yx}(x, y)$ to indicate that the first differentiation is with respect to y. Similarly, $\dfrac{\partial^2 f}{\partial y \partial x}$ is denoted by $f_{xy}(x, y)$.

Example 8

Find $f_{yx}(1,2)$ for the function $f(x,y) = x^3 + 2x^2y^2 + y^3$

Solution

$$f_x = 3x^2 + 4xy^2 \quad \text{and} \quad f_{yx} = 8xy \qquad \text{so} \quad f_{yx}(1,2) = 8 \times 1 \times 2 = 16.$$

Task

Find $f_{xx}(1,2),\ f_{yy}(-2,-1),\ f_{xy}(3,3)$ for $f(x,y) \equiv x^3 + 3x^2y^2 + y^2$.

Your solution

Answer

$f_x = 3x^2 + 6xy^2; \quad f_y = 6x^2y + 2y$

$f_{xx} = 6x + 6y^2; \quad f_{yy} = 6x^2 + 2; \quad f_{xy} = f_{yx} = 12xy$

$f_{xx}(1,2) = 6 + 24 = 30; \quad f_{yy}(-2,-1) = 26; \quad f_{xy}(3,3) = 108$

Engineering Example 1

The ideal gas law and Redlich-Kwong equation

Introduction

In Chemical Engineering it is often necessary to be able to equate the pressure, volume and temperature of a gas. One relevant equation is the ideal gas law

$$PV = nRT \qquad (1)$$

where P is pressure, V is volume, n is the number of moles of gas, T is temperature and R is the ideal gas constant ($= 8.314$ J mol^{-1} K^{-1}, when all quantities are in S.I. units). The ideal gas law has been in use since 1834, although its special cases at constant temperature (Boyle's Law, 1662) and constant pressure (Charles' Law, 1787) had been in use many decades previously.

While the ideal gas law is adequate in many circumstances, it has been superseded by many other laws where, in general, simplicity is weighed against accuracy. One such law is the Redlich-Kwong equation

$$P = \frac{RT}{V-b} - \frac{a}{\sqrt{T}\,V(V+b)} \qquad (2)$$

where, in addition to the variables in the ideal gas law, the extra parameters a and b are dependent upon the particular gas under consideration.

Clearly, in both equations the temperature, pressure and volume will be positive. Additionally, the Redlich-Kwong equation is only valid for values of volume greater than the parameter b - in practice however, this is not a limitation, since the gas would condense to a liquid before this point was reached.

Problem in words

Show that for both Equations (1) and (2)

(a) for constant temperature, the pressure decreases as the volume increases

(Note : in the Redlich-Kwong equation, assume that T is large.)

(b) for constant volume, the pressure increases as the temperature increases.

Mathematical statement of problem

For both Equations (1) and (2), and for the allowed ranges of the variables, show that

(a) $\dfrac{\partial P}{\partial V} < 0$ for $T =$ constant

(b) $\dfrac{\partial P}{\partial T} > 0$ for $V =$ constant

Assume that T is sufficiently large so that terms in $T^{-1/2}$ may be neglected when compared to terms in T.

Mathematical analysis

1. **Ideal gas law**
   This can be rearranged as

$$P = \frac{nRT}{V}$$

so that

(i) at constant temperature

$$\frac{\partial P}{\partial V} = \frac{-nRT}{V^2} \quad < 0 \quad \text{as all quantities are positive}$$

(ii) for constant volume

$$\frac{\partial P}{\partial T} = \frac{nR}{V} \quad > 0 \quad \text{as all quantities are positive}$$

2. **Redlich-Kwong equation**

$$\begin{aligned} P &= \frac{RT}{V-b} - \frac{a}{\sqrt{T}\,V(V+b)} \\ &= RT(V-b)^{-1} - a\,T^{-1/2}\,(V^2+Vb)^{-1} \end{aligned}$$

so that

(i) at constant temperature

$$\frac{\partial P}{\partial V} = -RT(V-b)^{-2} + a\,T^{-1/2}(V^2+Vb)^{-2}(2V+b)$$

which, for large T, can be approximated by

$$\frac{\partial P}{\partial V} \approx \frac{-RT}{(V-b)^2} \quad < 0 \quad \text{as all quantities are positive}$$

(ii) for constant volume

$$\frac{\partial P}{\partial T} = R(V-b)^{-1} + \frac{1}{2}\,a\,T^{-3/2}(V^2+Vb)^{-1} \quad > 0 \quad \text{as all quantities are positive}$$

Interpretation

In practice, the restriction on T is not severe, and regions in which $\dfrac{\partial P}{\partial V} < 0$ does not apply are those in which the gas is close to liquefying and, therefore, the entire Redlich-Kwong equation no longer applies.

Exercises

1. For the following functions find $\dfrac{\partial^2 f}{\partial x^2}, \dfrac{\partial^2 f}{\partial y^2}, \dfrac{\partial^2 f}{\partial x \partial y}, \dfrac{\partial^2 f}{\partial y \partial x}$.

   (a) $f(x,y) = x + 2y + 3$

   (b) $f(x,y) = x^2 + y^2$

   (c) $f(x,y) = x^3 + xy + y^3$

   (d) $f(x,y) = x^4 + xy^3 + 2x^3y^2$

   (e) $f(x,y,z) = xy + yz$

2. For the functions of Exercise 1 (a) to (d) find $f_{xx}(1,-3), \quad f_{yy}(-2,-2), \quad f_{xy}(-1,1)$.

3. For the following functions find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial^2 f}{\partial x \partial t}$

   (a) $f(x,t) = x\sin(tx) + x^2 t$ (b) $f(x,t,z) = zxt - e^{xt}$ (c) $f(x,t) = 3\cos(t + x^2)$

Answers

1. (a) $\dfrac{\partial^2 f}{\partial x^2} = 0 = \dfrac{\partial^2 f}{\partial y^2} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$

   (b) $\dfrac{\partial^2 f}{\partial x^2} = 2 = \dfrac{\partial^2 f}{\partial y^2}; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 0$

   (c) $\dfrac{\partial^2 f}{\partial x^2} = 6x, \quad \dfrac{\partial^2 f}{\partial y^2} = 6y; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 1.$

   (d) $\dfrac{\partial^2 f}{\partial x^2} = 12x^2 + 12xy^2, \quad \dfrac{\partial^2 f}{\partial y^2} = 6xy + 4x^3, \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 3y^2 + 12x^2 y$

   (e) $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial y^2} = 0; \quad \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x} = 1$

2.

	$f_{xx}(1,-3)$	$f_{yy}(-2,-2)$	$f_{xy}(-1,1)$
(a)	0	0	0
(b)	2	2	0
(c)	6	-12	1
(d)	120	-8	15

3. (a) $\dfrac{\partial f}{\partial x} = \sin(tx) + xt\cos(tx) + 2xt \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = 2x\cos(tx) - x^2 t\sin(tx) + 2x$

   (b) $\dfrac{\partial f}{\partial x} = zt - te^{xt} \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = z - e^{xt} - txe^{xt}$

   (c) $\dfrac{\partial f}{\partial x} = -6x\sin(t + x^2) \qquad \dfrac{\partial^2 f}{\partial t \partial x} = \dfrac{\partial^2 f}{\partial x \partial t} = -6x\cos(t + x^2)$

Stationary Points 18.3

Introduction

The calculation of the optimum value of a function of two variables is a common requirement in many areas of engineering, for example in thermodynamics. Unlike the case of a function of one variable we have to use more complicated criteria to distinguish between the various types of stationary point.

Prerequisites

Before starting this Section you should . . .

- understand the idea of a function of two variables
- be able to work out partial derivatives

Learning Outcomes

On completion you should be able to . . .

- identify local maximum points, local minimum points and saddle points on the surface $z = f(x, y)$
- use first partial derivatives to locate the stationary points of a function $f(x, y)$
- use second partial derivatives to determine the nature of a stationary point

1. The stationary points of a function of two variables

Figure 7 shows a computer generated picture of the surface defined by the function $z = x^3 + y^3 - 3x - 3y$, where both x and y take values in the interval $[-1.8, 1.8]$.

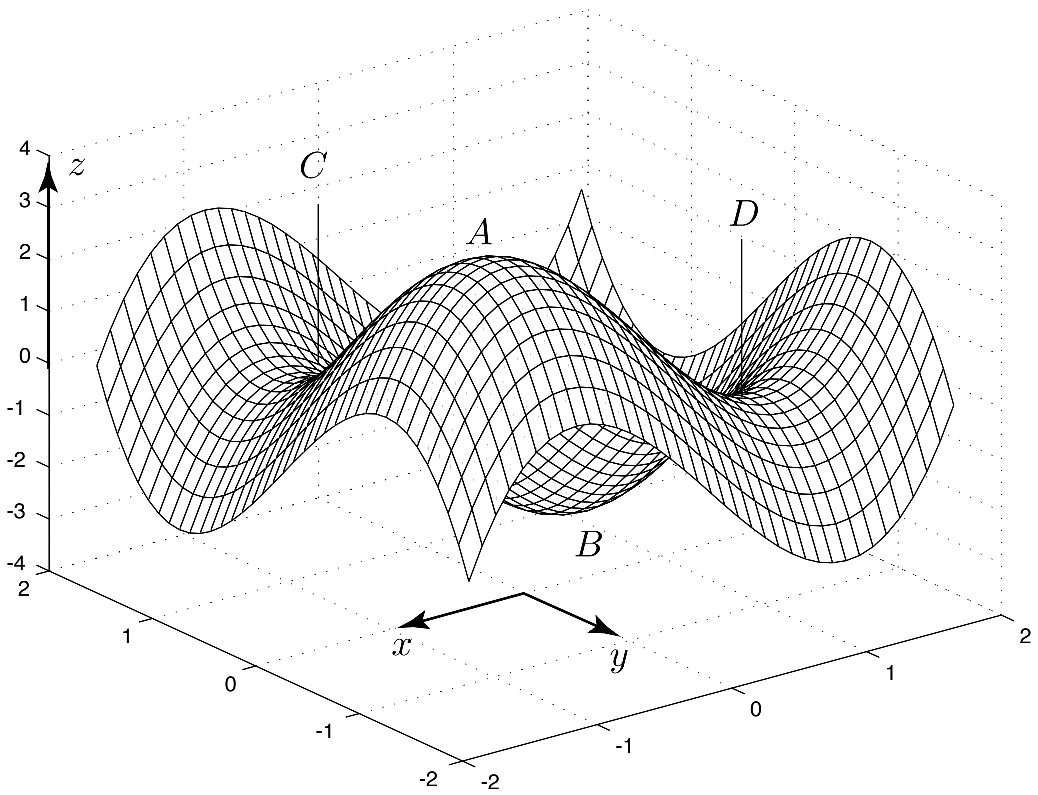


Figure 7

There are four features of particular interest on the surface. At point A there is a **local maximum**, at B there is a **local minimum**, and at C and D there are what are known as **saddle points**.

At A the surface is at its greatest height in the immediate neighbourhood. If we move on the surface from A we immediately lose height no matter in which direction we travel. At B the surface is at its least height in the neighbourhood. If we move on the surface from B we immediately gain height, no matter in which direction we travel.

The features at C and D are quite different. In some directions as we move away from these points along the surface we lose height whilst in others we gain height. The similarity in shape to a horse's saddle is evident.

At each point P of a *smooth* surface one can draw a unique plane which touches the surface there. This plane is called the **tangent plane** at P. (The tangent plane is a natural generalisation of the tangent line which can be drawn at each point of a smooth curve.) In Figure 7 at each of the points A, B, C, D the tangent plane to the surface is horizontal at the point of interest. Such points are thus known as **stationary points** of the function. In the next subsections we show how to locate stationary points and how to determine their nature using partial differentiation of the function $f(x, y)$,

Task

In Figures 8 and 9 what are the features at A and B?

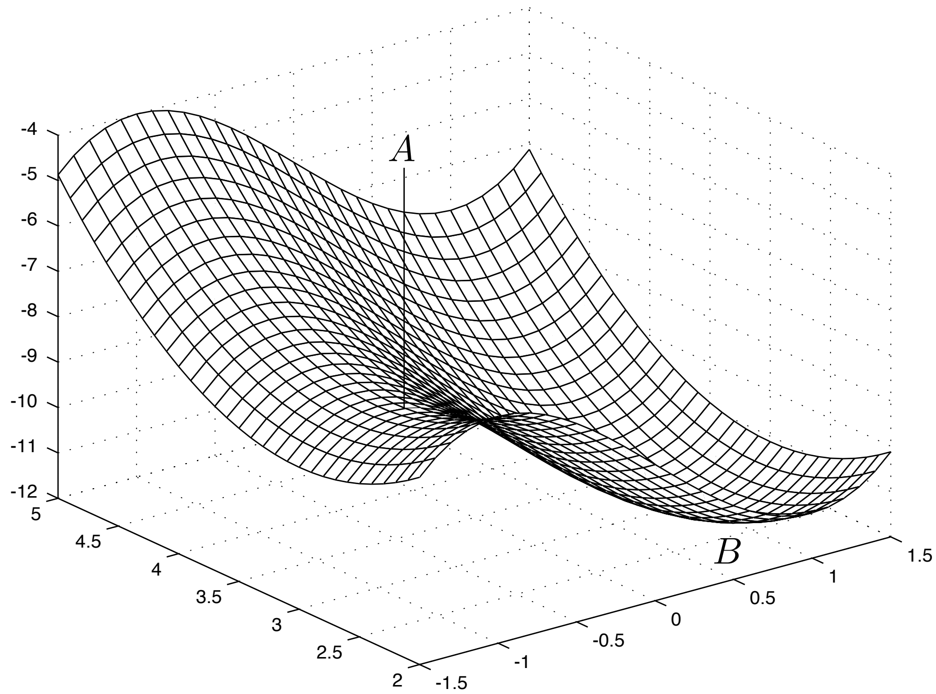


Figure 8

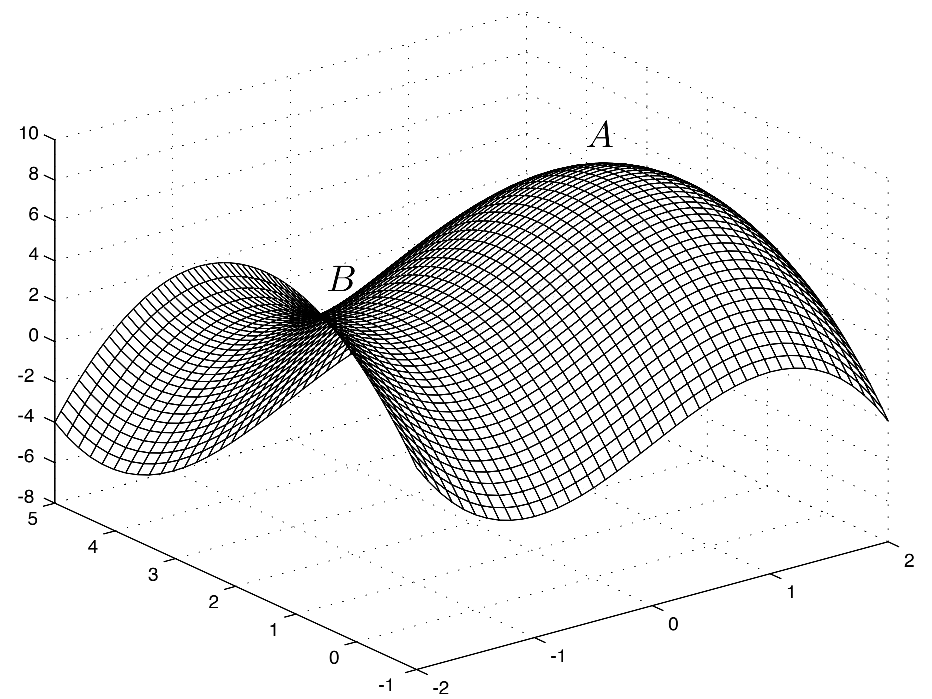


Figure 9

Your solution

Answer

Figure 8	A is a saddle point, B is a local minimum.
Figure 9	A is a local maximum, B is a saddle point.

2. Location of stationary points

As we said in the previous subsection, the tangent plane to the surface $z = f(x, y)$ is horizontal at a stationary point. A condition which guarantees that the function $f(x, y)$ will have a stationary point at a point (x_0, y_0) is that, at that point both $f_x = 0$ and $f_y = 0$ simultaneously.

Task

Verify that $(0, 2)$ is a stationary point of the function $f(x, y) = 8x^2+6y^2-2y^3+5$ and find the stationary value $f(0, 2)$.

First, find f_x and f_y:

Your solution

Answer

$f_x = 16x \ ; \quad f_y = 12y - 6y^2$

Now find the values of these partial derivatives at $x = 0,\ y = 2$:

Your solution

Answer

$f_x = 0\ , \quad f_y = 24 - 24 = 0$

Hence $(0, 2)$ is a stationary point.

The **stationary value** is $f(0, 2) = 0 + 24 - 16 + 5 = 13$

Example 9

Find a second stationary point of $f(x, y) = 8x^2 + 6y^2 - 2y^3 + 5$.

Solution

$f_x = 16x$ and $f_y \equiv 6y(2 - y)$. From this we note that $f_x = 0$ when $x = 0$, and $f_x = 0$ and when $y = 0$, so $x = 0,\ y = 0$ i.e. $(0, 0)$ is a second stationary point of the function.

It is important when solving the simultaneous equations $f_x = 0$ and $f_y = 0$ to find stationary points not to miss any solutions. A useful tip is to factorise the left-hand sides and consider systematically all the possibilities.

Example 10

Locate the stationary points of

$$f(x, y) = x^4 + y^4 - 36xy$$

Solution

First we write down the partial derivatives of $f(x, y)$

$$\frac{\partial f}{\partial x} = 4x^3 - 36y = 4(x^3 - 9y) \qquad \frac{\partial f}{\partial y} = 4y^3 - 36x = 4(y^3 - 9x)$$

Now we solve the equations $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$:

$$\begin{aligned} x^3 - 9y &= 0 && \text{(i)} \\ y^3 - 9x &= 0 && \text{(ii)} \\ \text{From (ii) we obtain:} \quad x &= \frac{y^3}{9} && \text{(iii)} \end{aligned}$$

Now substitute from (iii) into (i)

$$\begin{aligned} \frac{y^9}{9^3} - 9y &= 0 \\ \Rightarrow \qquad y^9 - 9^4 y &= 0 \\ \Rightarrow \qquad y(y^8 - 3^8) &= 0 \quad \text{(removing the common factor)} \\ \Rightarrow \qquad y(y^4 - 3^4)(y^4 + 3^4) &= 0 \quad \text{(using the difference of two squares)} \end{aligned}$$

We therefore obtain, as the only solutions:

$$y = 0 \text{ or } y^4 - 3^4 = 0 \quad \text{(since } y^4 + 3^4 \text{ is never zero)}$$

The last equation implies:

$$\begin{aligned} (y^2 - 9)(y^2 + 9) &= 0 \quad \text{(using the difference of two squares)} \\ \therefore \qquad y^2 = 9 \text{ and } y &= \pm\, 3. \end{aligned}$$

Now, using (iii): when $y = 0,\ x = 0$, when $y = 3,\ x = 3$, and when $y = -3,\ x = -3$.
The stationary points are $(0, 0),\ (-3, -3)$ and $(3, 3)$.

Task

Locate the stationary points of

$$f(x, y) = x^3 + y^2 - 3x - 6y - 1.$$

First find the partial derivatives of $f(x, y)$:

Your solution

Answer

$\dfrac{\partial f}{\partial x} = 3x^2 - 3, \quad \dfrac{\partial f}{\partial y} = 2y - 6$

Now solve simultaneously the equations $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$:

Your solution

Answer

$3x^2 - 3 = 0$ and $2y - 6 = 0$.

Hence $x^2 = 1$ and $y = 3$, giving stationary points at $(1, 3)$ and $(-1, 3)$.

3. The nature of a stationary point

We state, without proof, a relatively simple test to determine the nature of a stationary point, once located. If the surface is very flat near the stationary point then the test will not be sensitive enough to determine the nature of the point. The test is dependent upon the values of the second order derivatives: f_{xx}, f_{yy}, f_{xy} and also upon a combination of second order derivatives denoted by D where

$$D \equiv \frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$, which is also expressible as $D \equiv f_{xx}f_{yy} - (f_{xy})^2$

The test is as follows:

Key Point 4

Test to Determine the Nature of Stationary Points

1. At each stationary point work out the three second order partial derivatives.
2. Calculate the value of $D = f_{xx}f_{yy} - (f_{xy})^2$ at each stationary point.

   Then, test each stationary point in turn:
3. If $D < 0$ the stationary point is a **saddle point**.

   If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$ the stationary point is a **local minimum**.

   If $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2} < 0$ the stationary point is a **local maximum.**

   If $D = 0$ then the test is inconclusive (we need an alternative test).

Example 11

The function: $f(x,y) = x^4 + y^4 - 36xy$ has stationary points at $(0,0)$, $(-3,-3)$, $(3,3)$. Use Key Point 4 to determine the nature of each stationary point.

Solution

We have $\dfrac{\partial f}{\partial x} = f_x = 4x^3 - 36y$ and $\dfrac{\partial f}{\partial y} = f_y = 4y^3 - 36x$.

Then $\dfrac{\partial^2 f}{\partial x^2} = f_{xx} = 12x^2$, $\dfrac{\partial^2 f}{\partial y^2} = f_{yy} = 12y^2$, $\dfrac{\partial^2 f}{\partial x \partial y} = f_{yx} = -36$.

A tabular presentation is useful for calculating $D = f_{xx}f_{yy} - (f_{xy})^2$:

Derivatives	Point $(0,0)$	Point $(-3,-3)$	Point $(3,3)$
f_{xx}	0	108	108
f_{yy}	0	108	108
f_{xy}	-36	-36	-36
D	< 0	> 0	> 0

$(0,0)$ is a saddle point; $(-3,-3)$ and $(3,3)$ are both local minima.

Task

Determine the nature of the stationary points of $f(x,y) = x^3 + y^2 - 3x - 6y - 1$, which are $(1,3)$ and $(1,-3)$.

Write down the three second partial derivatives:

Your solution

Answer

$f_{xx} = 6x$, $f_{yy} = 2$, $f_{xy} = 0$.

Now complete the table below and determine the nature of the stationary points:

Your solution

	Point	Point
Derivatives	$(1,3)$	$(-1,3)$
f_{xx}		
f_{yy}		
f_{xy}		
D		

Answer

	Point	Point
Derivatives	$(1,3)$	$(-1,3)$
f_{xx}	6	-6
f_{yy}	2	2
f_{xy}	0	0
D	> 0	< 0

State the nature of each stationary point:

Your solution

Answer

$(1,3)$ is a local minimum; $(-1,3)$ is a saddle point.

For most functions the procedures described above enable us to distinguish between the various types of stationary point. However, note the following example, in which these procedures fail.

Given $f(x,y) = x^4 + y^4 + 2x^2y^2$.

$$\frac{\partial f}{\partial x} = 4x^3 + 4xy^2, \qquad \frac{\partial f}{\partial y} = 4y^3 + 4x^2y,$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 4y^2, \qquad \frac{\partial^2 f}{\partial y^2} = 12y^2 + 4x^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = 8xy$$

Location: The stationary points are located where $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial y} = 0$, that is, where $4x^3+4xy^2 = 0$ and $4y^3+4x^2y = 0$. A simple factorisation implies $4x(x^2+y^2) = 0$ and $4y(y^2+x^2) = 0$. The only solution which satisfies both equations is $x = y = 0$ and therefore the only stationary point is $(0,0)$.

Nature: Unfortunately, all the second partial derivatives are zero at $(0,0)$ and therefore $D = 0$, so the test, as described in Key Point 4, fails to give us the necessary information.
However, in this example it is easy to see that the stationary point is in fact a local minimum. This could be confirmed by using a computer generated graph of the surface near the point $(0,0)$. Alternatively, we observe $x^4 + y^4 + 2x^2y^2 \equiv (x^2 + y^2)^2$ so $f(x,y) \geq 0$, the only point where $f(x,y) = 0$ being the stationary point. This is therefore a local (and global) minimum.

Exercises

Determine the nature of the stationary points of the function in each case:

1. $f(x,y) = 8x^2 + 6y^2 - 2y^3 + 5$
2. $f(x,y) = x^3 + 15x^2 - 20y^2 + 10$
3. $f(x,y) = 4 - x^2 - xy - y^2$
4. $f(x,y) = 2x^2 + y^2 + 3xy - 3y - 5x + 8$
5. $f(x,y) = (x^2 + y^2)^2 - 2(x^2 - y^2) + 1$
6. $f(x,y) = x^4 + y^4 + 2x^2y^2 + 2x^2 + 2y^2 + 1$

Answers

1. $(0,0)$ local minimum, $(0,2)$ saddle point.
2. $(0,0)$ saddle point, $(-10,0)$ local maximum.
3. $(0,0)$ local maximum.
4. $(-1,3)$ saddle point.
5. $(0,0)$ saddle point, (1,0) local minimum, $(-1,0)$ local minimum.
6. $f(x,y) \equiv (x^2 + y^2 + 1)^2$, local minimum at $(0,0)$.

Errors and Percentage Change

18.4

Introduction

When one variable is related to several others by a functional relationship it is possible to estimate the percentage change in that variable caused by given percentage changes in the other variables. For example, if the values of the input variables of a function are measured and the measurements are in error, due to limits on the precision of measurement, then we can use partial differentiation to estimate the effect that these errors have on the forecast value of the output.

Prerequisites

Before starting this Section you should ...

- understand the definition of partial derivatives and be able to find them

Learning Outcomes

On completion you should be able to ...

- calculate small errors in a function of more than one variable
- calculate approximate values for absolute error, relative error and percentage relative error

1. Approximations using partial derivatives

Functions of two variables

We saw in HELM 16.5 how to expand a function of a single variable $f(x)$ in a Taylor series:

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{(x - x_0)^2}{2!}f''(x_0) + \ldots$$

This can be written in the following alternative form (by replacing $x - x_0$ by h so that $x = x_0 + h$):

$$f(x_0 + h) = f(x_0) + hf'(x_0) + \frac{h^2}{2!}f''(x_0) + \ldots$$

This expansion can be generalised to functions of two or more variables:

$$f(x_0 + h,\ y_0 + k) \simeq f(x_0, y_0) + hf_x(x_0, y_0) + kf_y(x_0, y_0)$$

where, assuming h and k to be small, we have ignored higher-order terms involving powers of h and k. We define δf to be the change in $f(x, y)$ resulting from small changes to x_0 and y_0, denoted by h and k respectively. Thus:

$$\delta f = f(x_0 + h,\ y_0 + k) - f(x_0, y_0)$$

and so $\delta f \simeq hf_x(x_0, y_0) + kf_y(x_0, y_0)$. Using the notation δx and δy instead of h and k for small increments in x and y respectively we may write

$$\delta f \simeq \delta x.f_x(x_0, y_0) + \delta y.f_y(x_0, y_0)$$

Finally, using the more common notation for partial derivatives, we write

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y.$$

Informally, the term δf is referred to as the **absolute error** in $f(x, y)$ resulting from errors $\delta x, \delta y$ in the variables x and y respectively. Other measures of error are used. For example, the **relative error** in a variable f is defined as $\dfrac{\delta f}{f}$ and the **percentage relative error** is $\dfrac{\delta f}{f} \times 100$.

Key Point 5

Measures of Error

If δf is the change in f at (x_0, y_0) resulting from small changes h, k to x_0 and y_0 respectively, then $\delta f = f(x_0 + h,\ y_0 + k) - f(x_0, y_0)$, and

The absolute error in f is δf.

The relative error in f is $\dfrac{\delta f}{f}$.

The percentage relative error in f is $\dfrac{\delta f}{f} \times 100$.

Note that to determine the error numerically we need to know not only the actual values of δx and δy but also the values of x and y at the point of interest.

Example 12

Estimate the absolute error for the function $f(x, y) = x^2 + x^3y$

Solution

$f_x = 2x + 3x^2y;\;\; f_y = x^3.$
Then $\delta f \simeq (2x + 3x^2y)\delta x + x^3\delta y$

Task

Estimate the absolute error for $f(x, y) = x^2y^2 + x + y$ at the point $(-1, 2)$ if $\delta x = 0.1$ and $\delta y = 0.025$. Compare the estimate with the exact value of the error.

First find f_x and f_y:

Your solution

$f_x =$ $\qquad\qquad$ $f_y =$

Answer

$f_x = 2xy^2 + 1, \quad f_y = 2x^2y + 1$

Now obtain an expression for the absolute error:

Your solution

Answer

$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2y + 1)\delta y$

Now obtain the estimated value of the absolute error at the point of interest:

Your solution

Answer

$\delta f \simeq (2xy^2 + 1)\delta x + (2x^2y + 1)\delta y = (-7)(0.1) + (5)(0.025) = -0.575.$

Finally compare the estimate with the exact value:

Your solution

Answer
The actual error is calculated from

$$\delta f = f(x_0 + \delta x,\ y_0 + \delta y) - f(x_0, y_0) = f(-0.9, 2.025) - f(-1, 2) = -0.5534937.$$

We see that there is a reasonably close correspondence between the two values.

Functions of three or more variables

If f is a function of several variables $x, y, u, v, \ldots$ the error induced in f as a result of making small errors $\delta x, \delta y, \delta u, \delta v \ldots$ in $x, y, u, v, \ldots$ is found by a simple generalisation of the expression for two variables given above:

$$\delta f \simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial u}\delta u + \frac{\partial f}{\partial v}\delta v + \ldots$$

Example 13

Suppose that the area of triangle ABC is to be calculated by measuring two sides and the included angle. Call the sides b and c and the angle A.
Then the area S of the triangle is given by $S = \frac{1}{2}bc\sin A.$

Now suppose that the side b is measured as 4.00 m, c as 3.00 m and A as 30^o. Suppose also that the measurements of the sides could be in error by as much as $\pm\ 0.005$ m and of the angle by $\pm\ 0.01^o$. Calculate the likely maximum error induced in S as a result of the errors in the sides and angle.

Solution

Here S is a function of three variables b, c, A. We calculate $S = \frac{1}{2} \times 4 \times 3 \times \frac{1}{2} = 3$ m^2.

Now $\frac{\partial S}{\partial b} = \frac{1}{2}c\sin A$, $\frac{\partial S}{\partial c} = \frac{1}{2}b\sin A$ and $\frac{\partial S}{\partial A} = \frac{1}{2}bc\cos A$, so

$$\delta S \simeq \frac{\partial S}{\partial b}\delta b + \frac{\partial S}{\partial c}\delta c + \frac{\partial S}{\partial A}\delta A = \frac{1}{2}c\sin A\ \delta b + \frac{1}{2}b\sin A\ \delta c + \frac{1}{2}bc\cos A\ \delta A.$$

Here $|\delta b|_{\max} = |\delta c|_{\max} = 0.005$ and $|\delta A|_{\max} = \frac{\pi}{180} \times 0.01$ (A must be measured in radians). Substituting these values we see that the maximum error in the calculated value of S is given by the approximation

$$\begin{aligned} |\delta S|_{\max} &\simeq \left(\frac{1}{2} \times 3 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times \frac{1}{2}\right) \times 0.005 + \left(\frac{1}{2} \times 4 \times 3 \times \frac{\sqrt{3}}{2}\right)\frac{\pi}{180} \times 0.01 \\ &\simeq 0.0097 \text{ m}^2 \end{aligned}$$

Hence the estimated value of S is in error by up to about $\pm\ 0.01$ m^2.

Engineering Example 2

Measuring the height of a building

The height h of a building is estimated from (i) the known horizontal distance x between the point of observation M and the foot of the building and (ii) the elevation angle θ between the horizontal and the line joining the point of observation to the top of the building (see Figure 10). If the measured horizontal distance is $x = 150$ m and the elevation angle is $\theta = 40°$, estimate the error in measured building height due to an error of 0.1° degree in the measurement of the angle of elevation.

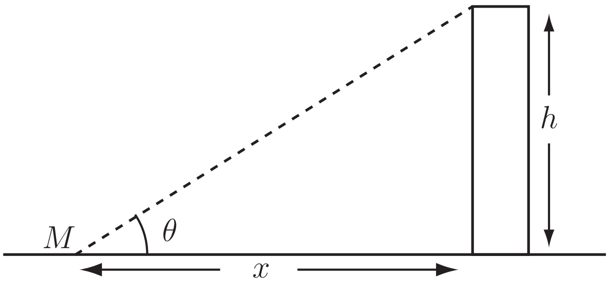


Figure 10: Geometry of the measurement

The variables $x,\ \theta$, and h are related by

$$\tan\theta = h/x.$$

or

$$x\tan\theta = h. \qquad (1)$$

The error in h resulting from a measurement error in θ can be deduced by differentiating (1):

$$\frac{d(x\tan\theta)}{d\theta} = \frac{dh}{d\theta} \qquad \Rightarrow \qquad \tan\theta\,\frac{dx}{d\theta} + x\,\frac{d(\tan\theta)}{d\theta} = \frac{dh}{d\theta}.$$

This can be written

$$\tan\theta\,\frac{dx}{d\theta} + x\sec^2\theta = \frac{dh}{d\theta}. \qquad (2)$$

Equation (2) gives the relationship among the small variations in variables x, h and θ. Since x is assumed to be without error and independent of θ, $\dfrac{dx}{d\theta} = 0$ and equation (2) becomes

$$x\sec^2\theta = \frac{dh}{d\theta}. \qquad (3)$$

Equation (3) can be considered to relate the error in building height δh to the error in angle $\delta\theta$:

$$\frac{\delta h}{\delta\theta} \simeq x\sec^2\theta.$$

It is given that $x = 150$ m.

The incidence angle $\theta = 40°$ can be converted to radians i.e. $\theta = 40\pi/180$ rad $= 2\pi/9$ rad.

Then the error in angle $\delta\theta = 0.1°$ needs to be expressed in radians for consistency of the units in (3).

So $\delta\theta = 0.1\pi/180$ rad $= \pi/1800$ rad. Hence, from Equation (3)

$$\delta h = 150\frac{\pi}{1800 \times \cos^2(2\pi/9)} \approx 0.45 \text{ m.}$$

So the error in building height resulting from an error in elevation angle of 0.1° is about 0.45 m.

Task

Estimate the maximum error in $f(x,y) = x^2 + y^2 + xy$ at the point $x = 2,\ y = 3$ if maximum errors $\pm\ 0.01$ and $\pm\ 0.02$ are made in x and y respectively.

First find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad\qquad$ $\dfrac{\partial f}{\partial y} =$

Answer

$\dfrac{\partial f}{\partial x} = 2x + y; \quad \dfrac{\partial f}{\partial y} = 2y + x.$

For $x = 2$ and $y = 3$ calculate the value of $f(x,y)$:

Your solution

Answer

$f(2,3) = 2^2 + 3^2 + 2 \times 3 = 19.$

Now since the error in the measured value of x is $\pm\ 0.01$ and in y is $\pm\ 0.02$ we have $|\delta x|_{\max} = 0.01, \quad |\delta y|_{\max} = 0.02$. Write down an expression to approximate to $|\delta f|_{\max}$:

Your solution

Answer

$|\delta f|_{\max} \simeq |(2x + y)|\ |\delta x|_{\max} + |(2y + x)|\ |\delta y|_{\max}$

Calculate $|\delta f|_{\max}$ at the point $x = 2,\ y = 3$ and give bounds for $f(2,3)$:

Your solution

Answer

$$\begin{aligned} |\delta f|_{\max} &\simeq (2 \times 2 + 3) \times 0.01 + (2 \times 3 + 2) \times 0.02 \\ &= 0.07 + 0.16 = 0.23. \end{aligned}$$

Hence we quote $f(2,3) = 19 \pm\ 0.23$, which can be expressed as $18.77 \leq f(2,3) \leq 19.23$

2. Relative error and percentage relative error

Two other measures of error can be obtained from a knowledge of the expression for the absolute error. As mentioned earlier, the relative error in f is $\dfrac{\delta f}{f}$ and the percentage relative error is $\left(\dfrac{\delta f}{f} \times 100\right)\%$. Suppose that $f(x,y) = x^2 + y^2 + xy$ then

$$\begin{aligned} \delta f &\simeq \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y \\ &= (2x+y)\delta x + (2y+x)\delta y \end{aligned}$$

The relative error is

$$\begin{aligned} \frac{\delta f}{f} &\simeq \frac{1}{f}\frac{\partial f}{\partial x}\delta x + \frac{1}{f}\frac{\partial f}{\partial y}\delta y \\ &= \frac{(2x+y)}{x^2+y^2+xy}\delta x + \frac{(2y+x)}{x^2+y^2+xy}\delta y \end{aligned}$$

The actual value of the relative error can be obtained if the actual errors of the independent variables are known and the values of x and y at the point of interest. In the special case where the function is a combination of powers of the input variables then there is a short cut to finding the relative error in the value of the function. For example, if $f(x,y,u) = \dfrac{x^2y^4}{u^3}$ then

$$\frac{\partial f}{\partial x} = \frac{2xy^4}{u^3}, \qquad \frac{\partial f}{\partial y} = \frac{4x^2y^3}{u^3}, \qquad \frac{\partial f}{\partial u} = -\frac{3x^2y^4}{u^4}$$

Hence

$$\delta f \simeq \frac{2xy^4}{u^3}\delta x + \frac{4x^2y^3}{u^3}\delta y - \frac{3x^2y^4}{u^4}\delta u$$

Finally,

$$\frac{\delta f}{f} \simeq \frac{2xy^4}{u^3} \times \frac{u^3}{x^2y^4}\delta x + \frac{4x^2y^3}{u^3} \times \frac{u^3}{x^2y^4}\delta y - \frac{3x^2y^4}{u^4} \times \frac{u^3}{x^2y^4}\delta u$$

Cancelling down the fractions,

$$\frac{\delta f}{f} \simeq 2\frac{\delta x}{x} + 4\frac{\delta y}{y} - 3\frac{\delta u}{u} \qquad (1)$$

so that

rel. error in $f \simeq 2\times$ (rel. error in x) $+ 4\times$ (rel. error in y) $- 3\times$ (rel. error in u).

Note that if we write

$$f(x,y,u) = x^2y^4u^{-3}$$

we see that the coefficients of the relative errors on the right-hand side of (1) are the powers of the appropriate variable.

To find the percentage relative error we simply multiply the relative error by 100.

Task

If $f = \dfrac{x^3y}{u^2}$ and x, y, u are subject to percentage relative errors of $1\%,\ -1\%$ and 2% respectively find the approximate percentage relative error in f.

First find $\dfrac{\partial f}{\partial x}$, $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial u}$:

Your solution

$\dfrac{\partial f}{\partial x} =$ $\qquad$ $\dfrac{\partial f}{\partial y} =$ $\qquad$ $\dfrac{\partial f}{\partial u} =$

Answer

$\dfrac{\partial f}{\partial x} = \dfrac{3x^2y}{u^2}, \quad \dfrac{\partial f}{\partial y} = \dfrac{x^3}{u^2}, \quad \dfrac{\partial f}{\partial u} = -\dfrac{2x^3y}{u^3}.$

Now write down an expression for δf

Your solution

$\delta f \simeq$

Answer

$\delta f \simeq \dfrac{3x^2y}{u^2}\delta x + \dfrac{x^3}{u^2}\delta y - \dfrac{2x^3y}{u^3}\delta u$

Hence write down an expression for the percentage relative error in f:

Your solution

Answer

$\dfrac{\delta f}{f} \times 100 \simeq \dfrac{3x^2y}{u^2} \times \dfrac{u^2}{x^3y}\delta x \times 100 + \dfrac{x^3}{u^2} \times \dfrac{u^2}{x^3y}\delta y \times 100 - \dfrac{2x^3y}{u^3} \times \dfrac{u^2}{x^3y}\delta u \times 100$

Finally, calculate the value of the percentage relative error:

Your solution

Answer

$$\begin{aligned}\frac{\delta f}{f} \times 100 &\simeq 3\frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 - 2\frac{\delta u}{u} \times 100 \\ &= 3(1) - 1 - 2(2) = -2\%\end{aligned}$$

Note that $f = x^3yu^{-2}$.

Engineering Example 3

Error in power to a load resistance

Introduction

The power required by an electrical circuit depends upon its components. However, the specification of the rating of the individual components is subject to some uncertainity. This Example concerns the calculation of the error in the power required by a circuit shown in Figure 11 given a formula for the power, the values of the individual components and the percentage errors in them.

Problem in words

The power delivered to the load resistance R_L for the circuit shown in Figure 11 is given by

$$P = \frac{25R_L}{(R+R_L)^2}$$

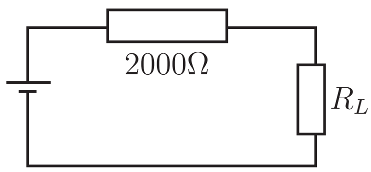


Figure 11: Circuit with a load resistance

If $R = 2000\ \Omega$ and $R_L = 1000\ \Omega$ with a maximum possible error of 5% in either, find P and estimate the maximum error in P.

Mathematical statement of the problem

We can calculate P by substituting $R = 2000$ and $R_L = 1000$ into $P = \dfrac{25R_L}{(R+R_L)^2}$.

We need to calculate the absolute errors in R and R_L and use these in the approximation $\delta P \approx \dfrac{P}{R}\delta R + \dfrac{P}{R_L}\delta R_L$ to calculate the error in P.

Mathematical analysis

At $R = 2000$ and $R_L = 1000$

$$P = \frac{25 \times 1000}{(1000+2000)^2} = \frac{25}{9000} = \frac{25}{9} \times 10^{-3} \approx 2.77 \times 10^{-3} \text{ watts.}$$

A 5% error in R gives $|\delta R|_{\max} = \dfrac{5}{100} \times 2000 = 100$ and $|\delta R_L|_{\max} = \dfrac{5}{100} \times 1000 = 50$

$$|\delta P|_{\max} \approx \frac{P}{R}\ |\delta R|_{\max} + \frac{P}{R_L}\ |\delta R_L|_{\max}$$

We need to calculate the values of the partial derivatives at $R = 2000$ and $R_L = 1000$.

$$P = \frac{25R_L}{(R+R_L)^2} = 25R_L(R+R_L)^{-2}$$

$$\frac{P}{R} = -50R_L(R+R_L)^{-3}$$

$$\frac{P}{R_L} = 25(R+R_L)^{-2} - 50R_L(R+R_L)^{-3}$$

So $\dfrac{P}{R}(2000, 1000) = -50(1000)(3000)^{-3} = \dfrac{-50}{1000^2 \times 27} = -\dfrac{50}{27} \times 10^{-6}$

$$\frac{P}{R_L}(2000, 1000) = 25(3000)^{-2} - 50(1000)(3000)^{-3} = \left(\frac{25}{9} - \frac{50}{27}\right) \times 10^{-6}$$

$$= \left(\frac{75-50}{27}\right) \times 10^{-6} = \frac{25}{27} \times 10^{-6}$$

Substituting these values into $|\delta P|_{\max} \approx \dfrac{P}{R}\ |\delta R|_{\max} + \dfrac{P}{R_L}\,|\delta R_L|_{\max}$ we get:

$$|\delta P|_{\max} = \frac{50}{27} \times 10^{-6} \times 100 + \frac{25}{27} \times 10^{-6} \times 50 = \left(\frac{5000}{27} + \frac{25 \times 50}{27}\right) \times 10^{-6} \approx 2.315 \times 10^{-4}$$

Interpretation

At $R = 2000$ and $R_L = 1000$, P will be 2.77×10^{-3} W and, assuming 5$\%$ errors in the values of the resistors, then the error in $P \approx \pm 2.315 \times 10^{-4}$ W. This represents about 8.4% error. So the error in the power is greater than that in the individual components.

Exercises

1. The sides of a right-angled triangle enclosing the right-angle are measured as 6 m and 8 m. The maximum errors in each measurement are $\pm$ 0.1m. Find the maximum error in the calculated area.

2. In Exercise 1, the angle opposite the 8 m side is calculated from $\tan\theta = 8/6$ as $\theta = 53°8'$. Calculate the approximate maximum error in that angle.

3. If $v = \sqrt{\dfrac{3x}{y}}$ find the maximum percentage error in v due to errors of 1% in x and 3% in y.

4. If $n = \dfrac{1}{2L}\sqrt{\dfrac{E}{d}}$ and L, E and d can be measured correct to within 1%, how accurate is the calculated value of n?

5. The area of a segment of a circle which subtends an angle θ is given by $A = \dfrac{1}{2}r^2(\theta - \sin\theta)$. The radius r is measured with a percentage error of $+0.2\%$ and θ is measured as 45^0 with an error of $= +0.1°$. Find the percentage error in the calculated area.

Answers

1. $A = \dfrac{1}{2}xy \qquad \delta A \approx \dfrac{\partial A}{\partial x}\delta x + \dfrac{\partial A}{\partial y}\delta y \qquad \delta A \approx \dfrac{y}{2}\delta x + \dfrac{x}{2}\delta y$

   Maximum error $= |y\delta x| + |x\delta y| = 0.7$ m^2.

2. $\theta = \tan^{-1}\dfrac{y}{x}$ so $\delta\theta = \dfrac{\partial \theta}{\partial x}\delta x + \dfrac{\partial \theta}{\partial y}\delta y = -\dfrac{y}{x^2+y^2}\delta x + \dfrac{x}{x^2+y^2}\delta y$

   Maximum error in θ is $\left|\dfrac{-8}{6^2+8^2}(0.1)\right| + \left|\dfrac{6}{6^2+8^2}(0.1)\right| = 0.014$ rad. This is 0.8^0.

3. Take logarithms of both sides: $\ln v = \dfrac{1}{2}\ln 3 + \dfrac{1}{2}\ln x - \dfrac{1}{2}\ln y$ so $\dfrac{\delta v}{v} \approx \dfrac{\delta x}{2x} - \dfrac{\delta y}{2y}$

   Maximum percentage error in $v = \left|\dfrac{\delta x}{2x}\right| + \left|-\dfrac{\delta y}{2y}\right| = \dfrac{1}{2}\% + \dfrac{3}{2}\% = 2\%$.

4. Take logarithms of both sides:

   $$\ln n = -\ln 2 - \ln L + \frac{1}{2}\ln E - \frac{1}{2}\ln d \qquad \text{so} \qquad \frac{\delta n}{n} = -\frac{\delta L}{L} + \frac{\delta E}{2E} - \frac{\delta d}{2d}$$

   Maximum percentage error in $n = \left|-\dfrac{\delta L}{L}\right| + \left|\dfrac{\delta E}{2E}\right| + \left|-\dfrac{\delta d}{2d}\right| = 1\% + \dfrac{1}{2}\% + \dfrac{1}{2}\% = 2\%$.

5. $A = \dfrac{1}{2}r^2(\theta - \sin\theta)$ so $\dfrac{\delta A}{A} = \dfrac{2\delta r}{r} + \dfrac{1-\cos\theta}{\theta - \sin\theta}\delta\theta$

   $$\frac{\delta A}{A} = 2(0.2)\% + \left\{\frac{1 - \dfrac{1}{\sqrt{2}}}{\dfrac{\pi}{4} - \dfrac{1}{\sqrt{2}}}\right\}\frac{\pi}{1800} \times 100\% = (0.4 + 0.65)\% = 1.05\%$$